# Measuring the dynamic viscosity of water at 20°C using Poiseuille's Law

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Using Poiseuille's Law, three values of the dynamic viscosity of water at 20.0°C are obtained. Measurements are taken by video analysis of the descent of water from a bottle connected to a capillary tube. The calculated values obtained are 1.69±0.04 mPa·s, 1.48±0.04 mPa·s and 1.49±0.04 mPa·s with the accepted value being 0.954±0.001 mPa·s. Chi-squared analysis is performed to show that the measurements are invalid. The discrepancy between measured and accepted values are due mostly to the large parallax error in the experimental setup. Problems with the experimental design and possible improvements to the method are also discussed.

## I. INTRODUCTION

Jean Louis Poiseuille, born 1799, was a French physician trained in physics and maths, motivated to understand blood flow through capillaries and veins, he developed what we now call Poiseuille's law which can be used to determine the viscosity of a fluid [1].

In this work, three values of the viscosity of water are obtained at 20.0°C through video analysis of the flow of water through a capillary tube. The values obtained are invalid due to an oversight in experimental design.

## II. THEORY

Laminar flow is characterised by adjacent layers of a flowing fluid sliding over each other with no mixing. When layers mix and vortices form, the flow is called turbulent. Dynamic viscosity (referred to in this report as viscosity) quantifies the resistance experienced by layers when sliding over each other in laminar flow. In a Newtonian fluid such as water, viscosity is independent of applied stress.

Poiseuille's law relates the flow rate $Q$ of a laminar Newtonian fluid through a long, thin, horizontal tube of length $L$ of constant radius $r$ to the pressure difference $\Delta p$ between the ends of the tube,

$$\Delta p = \frac{8\eta LQ}{\pi r^4} \tag{1}$$

with $\eta$ being the viscosity [1]. Connecting one end of the tube to an open container of water of constant cross sectional area $A$ and leaving the other end free to the atmosphere it can be shown using Eq.(1) that

$$h = h_0 e^{-\alpha t} \tag{2}$$

$$\text{with } \alpha = \frac{\pi \rho g r^4}{8\eta LA} \tag{3}$$

with $h$ being the height above the tube, $h_0$ being the initial height, $\alpha$ being the decay constant, $t$ being time, $\rho$ being the density of water and $g$ being local gravitational acceleration

The Reynolds number $Re$ indicates whether a fluids flow is laminar or turbulent, for the setup above it is given by [1]

$$Re = \frac{2Q\rho}{\eta\pi r} = \frac{2\alpha h_0 A e^{-\alpha t}\rho}{\eta\pi r} \tag{4}$$

. For laminar flow Re<2100 and for turbulent flow Re>4000 [1]. Measuring the height of the water line against time and fitting to

$$h = Ne^{-Mt} \tag{5}$$

by minimising the chi squared value (varying $N$ and $M$) allows the determination of the initial height and decay constant by comparison to Eq.(2). This can then be used to determine the viscosity by Eq.(3).

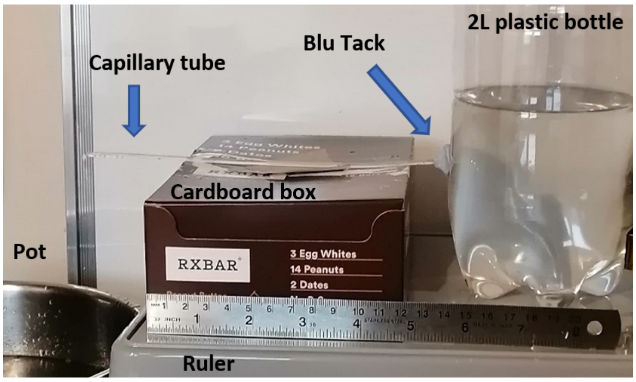


**FIG. 1:** Apparatus used to measure viscosity via Poiseuille's law.

## III. METHODS

The setup of the apparatus is shown in Figure 1. A small incision was made into a plastic bottle to fit a capillary tube and blu-tack was placed either side to minimise leaking. The capillary tube was fixed to a cardboard box to ensure that it was horizontal and straight to satisfy the conditions of Poiseuille's law.

The distance measurements required to obtain the constants in Eq.(3) were measured using the software GIMP. The dominant error in Eq.(3) is the radius of the capillary tube from the power of four dependence, the use of software minimises this error.

A video camera was used to film the descent of the water in the bottle. The height of the waterline was measured at equally spaced time intervals using the software Tracker. The experiments were performed in a well-lit space with a clear background to maximise the quality of video. The videos were stopped when droplets started forming at the capillary tube as this indicates the presence of external surface tension forces that are not accounted for in Eq.(1).

## IV. RESULTS

| Video | DoF | $\chi^2_{min}$ | $\chi^2_\nu$ | P($\chi^2_{min}$;$\nu$) |
|---|---|---|---|---|
| 1 | 229 | 1.021 | 0.00446 | 1.000 |
| 2 | 227 | 1.730 | 0.00762 | 1.000 |
| 3 | 212 | 1.488 | 0.00701 | 1.000 |

**TABLE I:** Goodness of fit statistics of the video data to the exponential model.

| Video | Decay constant (m·s$^{-1}$) | Viscosity (mPa·s) |
|---|---|---|
| 1 | 1.40 ± 0.03 | 1.69 ± 0.04 |
| 2 | 1.60 ± 0.03 | 1.48 ± 0.04 |
| 3 | 1.58 ± 0.03 | 1.49 ± 0.04 |
| Theoretical | 0.770 ± 0.008 | 0.954 ± 0.001 |

**TABLE II:** Results of measured values and theoretical prediction of water at 20.0°C.

The temperature of the water measured using a Garmin sports watch was $20.0 \pm 0.05$ °C. The evolution of the height of the waterline for the first video is shown in Figure 2 alongside the theoretical predicted value at the measured temperature. At this temperature the viscosity of water is given as $0.964 \pm 0.001$ mPa·s and density as $997.77 \pm 0.01$ kg·m$^{-3}$ [2]. The uncertainty in time and in the theoretical model are too small to be visible. The curve shown in the first video is similar for the other two repeats as shown by Figure 3 with the plots being separated for clarity.

The quality of fit of the minimised chi-squared model for each data set is shown in Table 1. From the decay constant of the fit model and Eq.(3), the value of the viscosity of water at 20.0°C has been calculated and compared to the predicted value, as shown in Table 2.

## V. DISCUSSION

The reduced chi-squared values shown in Tab.1 are significantly less than 1 showing that the exponential model Eq.(5) fits the data very well. This is visually confirmed in the subplot of Fig.2. However, these values are too small indicating that further investigation into errors is required.

The P values shown in Tab.1 are very close to 1, implying that the discrepancies between the model and data cannot explained by random statistical error. This indicates either overfitting of the model or overestimation of errors [3]. This is not due to overfitting as there are only two fit parameters and high degrees of freedom.

The main cause of the decreasing error bars in Fig.2 is parallax error. The point at which the water level is parallel to the camera is marked with a vertical line in Fig.3. The normalised residuals have peaks around this point as deviations become prominent for smaller error bars. Beyond these lines residuals seem offset from 0 , this could be due to Tracker measuring the water line at the back of the bottle.

All the normalised residuals in Fig.3 are less than one so are not normally distributed. The Durbin-Watson statistic $\mathcal{D}$ quantifies the non-random spread of normalised residuals [3]. For the first video, $\mathcal{D}$ is 0.061. This is close to 0, implying systematic correlation. This could be due to Tracker selecting the next measurement by searching in a small region around the current measurement, leading to non-independent errors.

Using Eq.(4) the maximum Reynolds number occurs at $t = 0$. For the first video this is $Re = 765 < 2100$, demonstrating the flow is laminar. Variation observed at the end of the residuals plots could be explained by surface tension inducing additional affect.

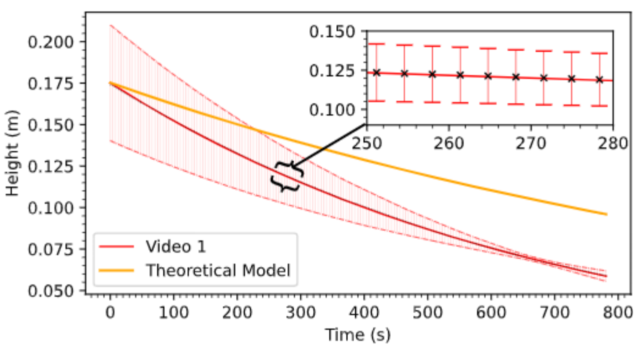


**FIG. 2:** Height drop of the waterline against predicted value at 20.0°C, the dotted lines indicate uncertainty at each point.

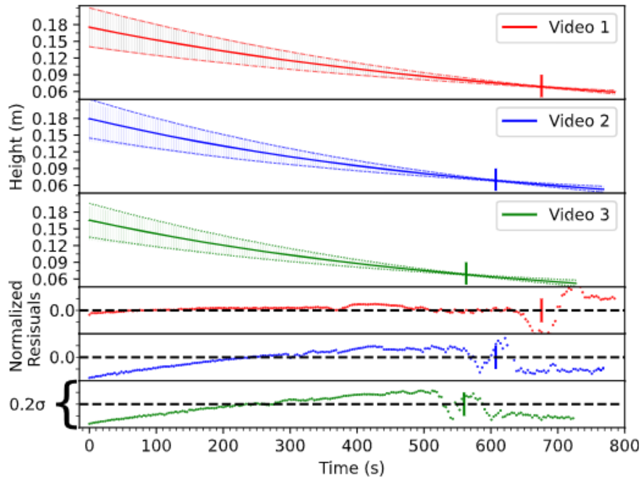


**FIG. 3:** Exponential decay of water height in the bottle with the zero parallax position marked, the normalised residual axis has height of 0.2 standard errors.

Visually, the theoretical value and the measured value do not agree as shown in Fig.2. From Tab.2 no measured result of the viscosity is consistent with the accepted value. The decay constants in Tab.2 have small errors despite the large visible error bar on each data point. This could be because the model is exponential and there are many data points, hence the chi-squared value only requires a small change to increase by one.

Although the sampling rate was increased by using video, the method introduced large parallax error. This invalidated the result of the decay parameter leading to unreliable values of viscosity of water. To minimise parallax, readings could have been taken by eye, parallel to the water level with a stopwatch. An alternative approach would be filming with a better-quality camera from a greater distance so parallax is reduced and the height could still be tracked accurately.

Further improvements to consider include the use of higher quality capillary tubes to ensure the diameter remains constant, and to prevent the formation of bubbles in the tube. Replacing the bottle with a large glass burette would ensure constant cross-sectional area, although the significance of the ridges in the bottle was not clear from the data. These are minor in effect in comparison to decreasing the error in parallax.

## VI. CONCLUSIONS

In this experiment, three values of the viscosity of water at 20°C were obtained using Poiseuille's Law and computerised tracking software. The measured data fit the exponential model well, however, large errors caused by parallax of the camera invalidate the values of viscosity obtained.

### References

[1] Frank M. White, *'Fluid Mechanics'*, 7th ed, The McGraw-Hill Companies Inc, Avenue of the Americas New York (2011).

[2] Eric W. Lemmon, Mark O. McLinden and Daniel G. Friend, *'Thermophysical Properties of Fluid Systems'*, NIST Chemistry WebBook, Database Number 69, https://doi.org/10.18434/T4D303 (retrieved November 24, 2020).

[3] I. G. Hughes and T. P. A. Hase, *'Measurements and their Uncertainties'*, Oxford University Press, Oxford (2010).

## VII. ERROR APPENDIX

**Uncertainty in height:** In the determination of the error of a single height measurement, it is assumed that the error in video time and calibration of the distance in Tracker are negligible. The uncertainty in time is based on the frame rate of the camera. The camera used filmed at 29.46 fps. The error of this is half the time of one frame giving 0.02 seconds, and the error from calibration would be less than a millimetre. On the scale of the height and time measurements, these can be ignored.

Parallax error was introduced as measurements were not made in the same plane as the water line. The parallax error $\alpha_{para}$ was calculated by considering the similar triangles formed by the setup as shown in Figure 4 giving:

$$\alpha_{para} = \frac{D_{bot} h_{scaled}}{L'} \tag{6}$$

where $L'$ is the distance from end of camera to start of bottle, $D_{bot}$ is the diameter of bottle and $h_{scaled}$ is the height from camera to waterline

The other two important errors were the uncertainty in water height position $\alpha_{pos}$ given by the standard error of multiple runs of the same video, and uncertainty in axis placement in Tracker $\alpha_{axis}$ given by 0.1 cm. The total error in height for a single measurement is thus [3]

$$\alpha_{height} = \sqrt{\alpha_{pos}^2 + \alpha_{axis}^2 + \alpha_{para}^2} \tag{7}$$

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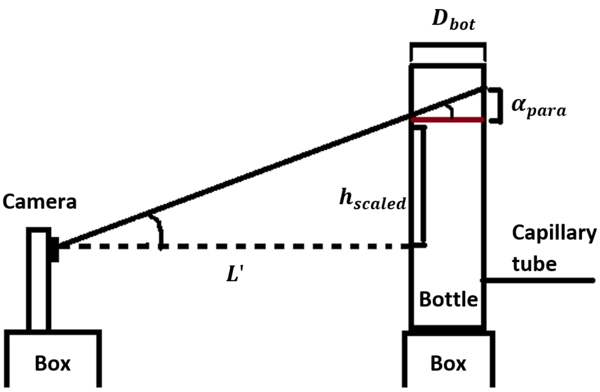


**FIG. 4:** A sketch of the setup showing parallax error.

**Uncertainty in viscosity:** The Nelder-Mead method of optimisation was used to obtain convergence of the minimised chi-squared parameters. Techniques based on gradients failed to converge, potentially due to the large number of data points coupled with the use of the exponential model.

To obtain errors on parameter values, a series of orthogonal steps were taken on the error surface as described in *'Measurements and Their Uncertainties'* Chapter 6 [3]. This gave the errors shown in Tab.2 which can be visually confirmed to be true by observation of the height of the $\chi^2_{min} + 1$ contour in Figure 5.

The error in tube length, tube radius and cross-section area of the bottle were calculated by taking the standard error of multiple readings. The constant radius of the tube was estimated by calculating half the average of the front and back diameters of the tube. The error in viscosity was then found by applying the functional method of error analysis to Eq.(3) [3].

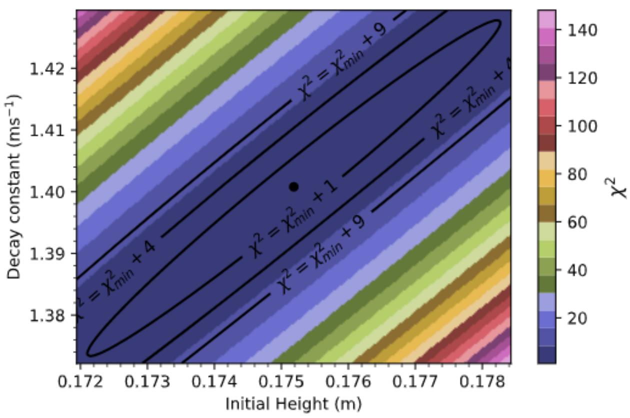


**FIG. 5:** Heat map of the $\chi^2$ function for video 1 with contours at 1,2,3 $\sigma$ away from the central dot at the minimum (0.175,1.40).